Zeta functions for locally symmetric spaces and graphs

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Introduction

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Riemann Hypothesis is one of the most famous and yet unsolved Clay Millennium prize problem: ...(a question about when is $\zeta(s) = 0$)

Analogy between arithmetic and geometry :

$$\text{prime numbers} \Longleftrightarrow \text{prime geodesics}$$

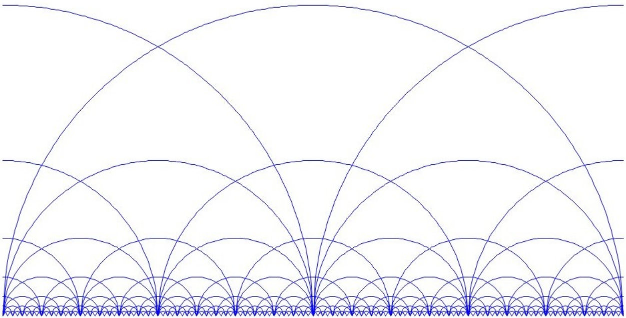
Selberg: A prime geodesic theorem for hyperbolic surfaces of finite volume.

The number $\pi_\Gamma(x)$ of classes of closed geodesics on $\Gamma\backslash\mathcal{H}$ whose length is $\leq x$ is given by:

$$\pi_\Gamma(x) = \int_0^x \frac{dt}{\log t} + E(x)$$

where the Error term is known to be $\ll x$ and believed to be $\sim x^{1/2}$.

Geodesics on hyperbolic space



Geodesics in the two-dimensional hyperbolic space

A theorem on length spectrum

Consider a compact hyperbolic space X_Γ of dimension n. Its length spectrum is defined as the number of homotopy classes of closed geodesics in X_Γ. Two lattices Γ_1 and Γ_2 are called length-isospectral if their length spectra are equal.

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An analogue for length spectra of classical strong multiplicity one theorem:

Theorem (- & Rajan, 2011)

Let $n \geq 2$ be even and Γ_1, Γ_2 be torsion-free uniform lattices in $SO(n,1)$. Suppose there exists a finite subset S of $[0,\infty)$ such that for every $\ell \notin S$, the numbers of homotopy classes of closed geodesics in $\Gamma_i \backslash SO(n,1)/SO(n)$ are equal for $i = 1,2$. Then the spaces $\Gamma_1 \backslash SO(n,1)/SO(n)$ and $\Gamma_2 \backslash SO(n,1)/SO(n)$ are length-isospectral.

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Key ideas: We used the analytic properties (analytic continuation and functional equation) of Ruelle Zeta function for compact hyperbolic spaces.

Zeta functions associated to a compact hyperbolic space

Ruelle Zeta function:

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This function 'stores' the geometric information (lengths of prime geodesics) of the space. This is an analogue of the Riemann Zeta function for hyperbolic spaces and has interesting analytical properties viz. an analytic continuation and functional equation on the plane $\mathbb{C}$.

Cycles on graphs as analogues of geodesics

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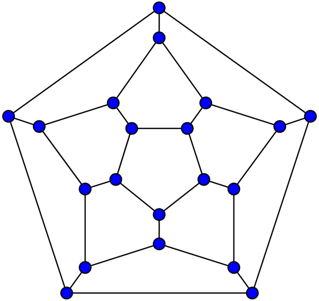
Analogy: A geometric realisation of the graph is like an 'one dimensional manifold'.

An example:



Zeta functions associated to graphs

For certain periodic regular graphs, the following Ihara Zeta function can be defined and it has nice analytic properties.

$$Z_{X,\Gamma}(s) := \prod_{[C]_\Gamma \in [\mathcal{P}]_\Gamma} (1 - s^{\ell(C)})^{-\frac{1}{|\Gamma_C|}},$$

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In a recent joint work with my Phd student Ayesha Fatima, we have proved an analogue of 'multiplicity one' theorem for the length spectrum of certain families of periodic graphs.

Current and future work

- Classical results in number theory: Prime number theorem and some similar results which describe the distribution of primes among all integers.
- Similar results have been established by works of Selberg, Sarnak, Wakayma and Kelmer among many others for locally symmetric spaces.
- We aim to study the asymptotic behaviour for cycles in periodic graphs.

Relation with spectral theory of graphs

- One associates certain linear operators to graphs which are analogues of the well known Laplace operator in geometry.
- In compact hyperbolic spaces, it is known that the spectral properties of such operators determine the behaviour of the geodesics.
- It will be interesting to know whether such a result holds for graphs (ongoing work with Ayesha Fatima) .

Thank you!