Primes: Dooriyan Nazdeekiyan Ban Gayi?

Kaneenika Sinha

Science at IISER Pune

27 December, 2017

Distribution of prime numbers

Distribution of prime numbers

- A prime number is a natural number > 1 which is not divisible by any natural number other than one and itself.

Distribution of prime numbers

- A prime number is a natural number > 1 which is not divisible by any natural number other than one and itself.
- Any natural number can be written uniquely as a product of powers of primes [Fundamental Theorem of Arithmetic].

Distribution of prime numbers

- A prime number is a natural number > 1 which is not divisible by any natural number other than one and itself.
- Any natural number can be written uniquely as a product of powers of primes [Fundamental Theorem of Arithmetic].
- How are prime numbers distributed on the number line?

Distribution of prime numbers

- A prime number is a natural number > 1 which is not divisible by any natural number other than one and itself.
- Any natural number can be written uniquely as a product of powers of primes [Fundamental Theorem of Arithmetic].
- How are prime numbers distributed on the number line? As a first step, we ask the value of $\pi(N)$, that is, how many prime numbers we can expect to have up to a large number N.

Distribution of prime numbers

- A prime number is a natural number > 1 which is not divisible by any natural number other than one and itself.
- Any natural number can be written uniquely as a product of powers of primes [Fundamental Theorem of Arithmetic].
- How are prime numbers distributed on the number line? As a first step, we ask the value of $\pi(N)$, that is, how many prime numbers we can expect to have up to a large number N.
- The Prime Number Theorem is the assertion that

$$\lim_{N\to\infty} \frac{\pi(N)\log N}{N} = 1.$$

Distribution of prime numbers

- A prime number is a natural number > 1 which is not divisible by any natural number other than one and itself.
- Any natural number can be written uniquely as a product of powers of primes [Fundamental Theorem of Arithmetic].
- How are prime numbers distributed on the number line? As a first step, we ask the value of $\pi(N)$, that is, how many prime numbers we can expect to have up to a large number N.
- The Prime Number Theorem is the assertion that

$$\lim_{N\to\infty} \frac{\pi(N)\log N}{N} = 1.$$

That is,

$$\pi(N) \sim \frac{N}{\log N} \text{ as } N \to \infty.$$

- If we arrange the primes in ascending order,

$$p_1 < p_2 < p_3 \cdots < p_n \ldots,$$

then the n-th prime number, $p_n \sim n \log n$.

- If we arrange the primes in ascending order,

$$p_1 < p_2 < p_3 \cdots < p_n \ldots,$$

then the n-th prime number, $p_n \sim n \log n$.

- To construct a 200-digit prime number, we randomly generate 200-digit numbers and test whether they are prime.

- If we arrange the primes in ascending order,

$$p_1 < p_2 < p_3 \cdots < p_n \ldots,$$

then the n-th prime number, $p_n \sim n \log n$.

- To construct a 200-digit prime number, we randomly generate 200-digit numbers and test whether they are prime. The prime number theorem says that after an average of 230 tries, we should obtain a prime number.

- If we arrange the primes in ascending order,

$$p_1 < p_2 < p_3 \cdots < p_n \ldots,$$

then the n-th prime number, $p_n \sim n \log n$.

- To construct a 200-digit prime number, we randomly generate 200-digit numbers and test whether they are prime. The prime number theorem says that after an average of 230 tries, we should obtain a prime number.
- In 1859, Riemann related the prime number theorem to the location of the zeros of the Riemann zeta function on the complex plane.

- If we arrange the primes in ascending order,

$$p_1 < p_2 < p_3 \cdots < p_n \ldots,$$

then the n-th prime number, $p_n \sim n \log n$.

- To construct a 200-digit prime number, we randomly generate 200-digit numbers and test whether they are prime. The prime number theorem says that after an average of 230 tries, we should obtain a prime number.
- In 1859, Riemann related the prime number theorem to the location of the zeros of the Riemann zeta function on the complex plane.
- The Riemann hypothesis predicts the relative error between the prime counting function $\pi(N)$ and $\frac{N}{\log N}$ for very large values of N.

- If we arrange the primes in ascending order,

$$p_1 < p_2 < p_3 \cdots < p_n \dots ,$$

then the n-th prime number, $p_n \sim n \log n$.

- To construct a 200-digit prime number, we randomly generate 200-digit numbers and test whether they are prime. The prime number theorem says that after an average of 230 tries, we should obtain a prime number.
- In 1859, Riemann related the prime number theorem to the location of the zeros of the Riemann zeta function on the complex plane.
- The Riemann hypothesis predicts the relative error between the prime counting function $\pi(N)$ and $\frac{N}{\log N}$ for very large values of N.
- Fundamental tool in analytic number theory: we convert counting problems into analysis problems. Counting problems in number sequences are studied by investigating analogues of the zeta function called L-functions.

Gaps between consecutive primes

- Finer question: How close or how far apart are consecutive primes?

Gaps between consecutive primes

- Finer question: How close or how far apart are consecutive primes?
- Note: we have arbitrarily long sequences of consecutive nonprime numbers.

Gaps between consecutive primes

- Finer question: How close or how far apart are consecutive primes?
- Note: we have arbitrarily long sequences of consecutive nonprime numbers.

$$n! + 2,\ n! + 3, \ldots n! + n.$$

Gaps between consecutive primes

- Finer question: How close or how far apart are consecutive primes?
- Note: we have arbitrarily long sequences of consecutive nonprime numbers.

$$n! + 2,\ n! + 3, \ldots n! + n.$$

- On the other hand, we have several pairs of twin primes $p,\ p+2$, [Open problem: are there infinitely many twin primes?]

Gaps between consecutive primes

- Finer question: How close or how far apart are consecutive primes?
- Note: we have arbitrarily long sequences of consecutive nonprime numbers.

$$n! + 2,\ n! + 3, \ldots n! + n.$$

- On the other hand, we have several pairs of twin primes p, $p + 2$, [Open problem: are there infinitely many twin primes?]
- Yitang Zhang: There are infinitely many pairs of consecutive primes with gaps < 70 million.

Gaps between consecutive primes

- Finer question: How close or how far apart are consecutive primes?
- Note: we have arbitrarily long sequences of consecutive nonprime numbers.

$$n! + 2,\ n! + 3, \ldots n! + n.$$

- On the other hand, we have several pairs of twin primes p, $p + 2$, [Open problem: are there infinitely many twin primes?]
- Yitang Zhang: There are infinitely many pairs of consecutive primes with gaps < 70 million. The gap has now been reduced to 246 through a Polymath project.

Spacings between primes

So, it makes sense to ask the question on average: PNT tells us

$$\frac{1}{N}\sum_{n\leq N}\frac{p_{n+1}-p_n}{\log p_n}\sim 1 \text{ as } N\to\infty.$$

Spacings between primes

So, it makes sense to ask the question on average: PNT tells us

$$\frac{1}{N}\sum_{n\le N}\frac{p_{n+1}-p_n}{\log p_n}\sim 1 \text{ as } N\to\infty.$$

Conjecture

The distribution of the normalized spacings between primes,

$$\frac{p_{n+1}-p_n}{\log p_n}$$

is Poissonnian. That is,

Spacings between primes

So, it makes sense to ask the question on average: PNT tells us

$$\frac{1}{N}\sum_{n\leq N}\frac{p_{n+1}-p_n}{\log p_n}\sim 1 \text{ as } N\to\infty.$$

Conjecture

The distribution of the normalized spacings between primes,

$$\frac{p_{n+1}-p_n}{\log p_n}$$

is Poissonnian. That is,
for any compactly supported, smooth function f on $[0,\infty)$,

$$\frac{1}{N}\sum_{n=1}^{N} f\left(\frac{p_{n+1}-p_n}{\log p_n}\right)\sim\int_0^\infty f(t)e^{-t}dt.$$

The τ-function of Ramanujan

In 1916, Srinivasa Ramanujan studied the arithmetic function $\tau : \mathbb{N} \to \mathbb{Z}$ defined by the following generating series

$$\Delta(x) := \sum_{n=1}^{\infty} \tau(n) x^n = x \prod_{n=1}^{\infty} (1 - x^n)^{24}, \ |x| < 1.$$

The τ-function of Ramanujan

In 1916, Srinivasa Ramanujan studied the arithmetic function $\tau : \mathbb{N} \to \mathbb{Z}$ defined by the following generating series

$$\Delta(x) := \sum_{n=1}^{\infty} \tau(n) x^n = x \prod_{n=1}^{\infty} (1 - x^n)^{24}, \ |x| < 1.$$

Ramanujan obtained the values of $\tau(n)$ up to $n = 30$ and conjectured the following three interesting properties of these coefficients:

The τ-function of Ramanujan

In 1916, Srinivasa Ramanujan studied the arithmetic function $\tau : \mathbb{N} \to \mathbb{Z}$ defined by the following generating series

$$\Delta(x) := \sum_{n=1}^{\infty} \tau(n) x^n = x \prod_{n=1}^{\infty} (1 - x^n)^{24}, \ |x| < 1.$$

Ramanujan obtained the values of $\tau(n)$ up to $n = 30$ and conjectured the following three interesting properties of these coefficients:

1. $\tau(mn) = \tau(m)\tau(n)$, if m and n are coprime.

The τ-function of Ramanujan

In 1916, Srinivasa Ramanujan studied the arithmetic function $\tau : \mathbb{N} \to \mathbb{Z}$ defined by the following generating series

$$\Delta(x) := \sum_{n=1}^{\infty} \tau(n) x^n = x \prod_{n=1}^{\infty} (1 - x^n)^{24}, \ |x| < 1.$$

Ramanujan obtained the values of $\tau(n)$ up to $n = 30$ and conjectured the following three interesting properties of these coefficients:

1. $\tau(mn) = \tau(m)\tau(n)$, if m and n are coprime.
2. For a prime p and $r \geq 1$, $\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1})$.

The τ-function of Ramanujan

In 1916, Srinivasa Ramanujan studied the arithmetic function $\tau : \mathbb{N} \to \mathbb{Z}$ defined by the following generating series

$$\Delta(x) := \sum_{n=1}^{\infty} \tau(n) x^n = x \prod_{n=1}^{\infty} (1 - x^n)^{24}, \ |x| < 1.$$

Ramanujan obtained the values of $\tau(n)$ up to $n = 30$ and conjectured the following three interesting properties of these coefficients:

1. $\tau(mn) = \tau(m)\tau(n)$, if m and n are coprime.
2. For a prime p and $r \geq 1$, $\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1})$.
3. $|\tau(p)| \leq 2p^{11/2}$.

Distribution of values of τ-function

- The first two conjectures were proved by Mordell in 1920.

Distribution of values of τ-function

- The first two conjectures were proved by Mordell in 1920.
- The third conjecture was proved by Deligne in 1974 using very deep ideas from arithmetic algebraic geometry.

Distribution of values of τ-function

- The first two conjectures were proved by Mordell in 1920.
- The third conjecture was proved by Deligne in 1974 using very deep ideas from arithmetic algebraic geometry.
- The distribution of

$$\frac{\tau(p)}{p^{11/2}}$$

was conjectured by Sato and Tate in 1960s.

Distribution of values of τ-function

- The first two conjectures were proved by Mordell in 1920.
- The third conjecture was proved by Deligne in 1974 using very deep ideas from arithmetic algebraic geometry.
- The distribution of
$$\frac{\tau(p)}{p^{11/2}}$$
was conjectured by Sato and Tate in 1960s. For any interval $[a, b] \subset [-2, 2]$,
$$\frac{1}{\pi(x)} \# \left\{ p \leq x : \frac{\tau(p)}{p^{11/2}} \in [a, b] \right\} \sim \int_a^b \frac{1}{\pi} \sqrt{\left(1 - \frac{x^2}{4}\right)} dx.$$

Distribution of values of τ-function

- The first two conjectures were proved by Mordell in 1920.
- The third conjecture was proved by Deligne in 1974 using very deep ideas from arithmetic algebraic geometry.
- The distribution of

$$\frac{\tau(p)}{p^{11/2}}$$

was conjectured by Sato and Tate in 1960s. For any interval $[a, b] \subset [-2, 2]$,

$$\frac{1}{\pi(x)} \# \left\{ p \leq x : \frac{\tau(p)}{p^{11/2}} \in [a, b] \right\} \sim \int_a^b \frac{1}{\pi} \sqrt{\left(1 - \frac{x^2}{4}\right)} dx.$$

- Proved by Richard Taylor in 2008 and in full generality by Barnet-Lamb, Geraghty, Harris and Taylor in 2011.

Distribution of values of τ-function

- The first two conjectures were proved by Mordell in 1920.
- The third conjecture was proved by Deligne in 1974 using very deep ideas from arithmetic algebraic geometry.
- The distribution of
$$\frac{\tau(p)}{p^{11/2}}$$
was conjectured by Sato and Tate in 1960s. For any interval $[a, b] \subset [-2, 2]$,
$$\frac{1}{\pi(x)} \# \left\{ p \leq x : \frac{\tau(p)}{p^{11/2}} \in [a, b] \right\} \sim \int_a^b \frac{1}{\pi} \sqrt{\left(1 - \frac{x^2}{4}\right)} dx.$$
- Proved by Richard Taylor in 2008 and in full generality by Barnet-Lamb, Geraghty, Harris and Taylor in 2011.
- This distribution property is closely linked with the behaviour of L-functions linked to $\Delta(x)$.

Spacings between $\tau(p)$'s: Joint work with B. Balasubramanyam

Spacings between $\tau(p)$'s: Joint work with B. Balasubramanyam

- We put $\tau(p) = 2\cos\pi\,\phi_\tau(p)$, with $\phi_\tau(p) \in [0,1]$.

Spacings between $\tau(p)$'s: Joint work with B. Balasubramanyam

- We put $\tau(p) = 2\cos\pi\,\phi_\tau(p)$, with $\phi_\tau(p) \in [0,1]$. We consider the sequence $\{\phi_\tau(p)\}_{p\to\infty}$ lying in $[0,1]$ and arrange

$$0 \leq (\phi_\tau(p))_1 \leq (\phi_\tau(p))_2 \leq (\phi_\tau(p))_3 \leq \dots$$

and investigate spacings between consecutive elements.

Spacings between $\tau(p)$'s: Joint work with B. Balasubramanyam

- We put $\tau(p) = 2\cos\pi\,\phi_\tau(p)$, with $\phi_\tau(p) \in [0,1]$. We consider the sequence $\{\phi_\tau(p)\}_{p\to\infty}$ lying in $[0,1]$ and arrange

$$0 \le (\phi_\tau(p))_1 \le (\phi_\tau(p))_2 \le (\phi_\tau(p))_3 \le \dots$$

  and investigate spacings between consecutive elements.
- $\Delta(x)$ is an example of a special class of modular cusp forms of weight 12 called Hecke eigenforms. One can consider variants Hecke eigenforms f of even, positive weights k.

- Each such f has a Fourier expansion of the form

$$f(z) := \sum_{n=1}^{\infty} n^{\frac{k-1}{2}} a_f(n) q^n, \; q = e^{2\pi i z}.$$

- Each such f has a Fourier expansion of the form

$$f(z) := \sum_{n=1}^{\infty} n^{\frac{k-1}{2}} a_f(n) q^n, \; q = e^{2\pi i z}.$$

- We write $a_f(p) = 2\cos \pi\phi_f(p)$, $\phi_f(p) \in [0,1]$ and order

$$0 \leq \phi_f(p)_1 \leq \phi_f(p)_2 \cdots \leq 1.$$

Theorem (B. Balasubramanyam, K.S.)

Let us consider families of Hecke eigenforms $\mathcal{F}_k$ with weights k sufficiently large. Then, as $N \to \infty$, for almost every Hecke eigenform $f \in \mathcal{F}_k$, the consecutive spacings between $\frac{p_n}{(\log p_n)^2}\phi_f(p_n)$ follow the Poissonnian distribution with parameter $\lambda = 1.5$.